INTERNAL FRICTION DESCRIBED WITH THE AID OF FRACTIONALLY-EXPONENTIAL KERNELS

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 103-106, 1966

Hereditary properties of solids undergoing shear deformation can be allowed for with the aid of one of the following equivalent expressions for the integral operator of the shear modulus:

$$\mu^*\varepsilon = \mu_0\varepsilon + \Delta\mu \int_{-\infty}^{t} \varphi(t-t')\,\varepsilon^{\cdot}(t')\,dt',$$

$$\mu^*\varepsilon = \mu_\infty\varepsilon - \Delta\mu \int_{-\infty}^{t} f(t-t')\,\varepsilon(t')\,dt' \qquad (1)$$

$$(\Delta\mu = \mu_\infty - \mu_0), \qquad (2)$$

where ε is the shear strain; μ_0 and μ_∞ are, respectively, the relaxed and nonrelaxed values of shear modulus; and φ and f are memory functions. Choice of the explicit form of these functions, which are kernels of integral operators, determines the behavior of linear viscoelastic media. The function φ has the following asymptotic properties: $\varphi(\infty) = 0$ and $\varphi(0) = 1$. The first of these reflects the requirement of finite relaxation times; the validity of the second can be easily verified by taking the strain rate to be impulsive: $\varepsilon(t) = \varepsilon_0\,\delta(t)$. With respect to function $f(t)$ it is necessary to stipulate $f(\infty) = 0$, but the condition of finiteness of $f(t)$ as $t \to 0$ is not directly imposed by Eq. (2). Numerous studies of stress relaxation and elastic aftereffect in solids show that mechanical relaxation can be satisfactorily described only by considering memory functions $f(t)$ which have singularities as $t \to 0$ [1]. The most convenient for practical calculations are the memory functions postulated by Rabotnov [2],

$$f(t) = \frac{1}{\tau}\left(\frac{t}{\tau}\right)^{\gamma-1} \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau)^{\gamma n}}{\Gamma[\gamma(n+1)]} \qquad (0<\gamma<1). \quad (3)$$

Here, together with the usual notation, the following substitution was made: $\beta \to -\tau^{-\gamma}$ and $\alpha \to \gamma - 1$.

In the case of periodic deformation $\varepsilon = \varepsilon_0 \exp i\omega t$, Rabotnov's kernel leads to the following expression for the complex elastic modulus:

$$\mu(\omega) = \mu_\infty - \frac{\mu_\infty - \mu_0}{1 + (i\omega\tau)^\gamma}. \qquad (4)$$

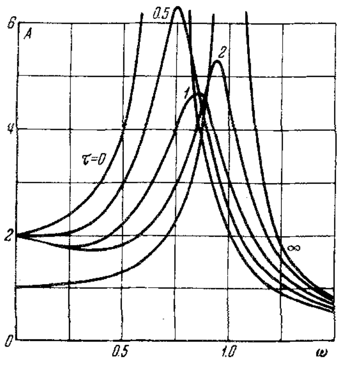

Fig. 1

Taking into account that the memory function $\varphi(t)$, which determines the integral operator (1), is related to the distribution function Φ of relaxation frequencies $s = 1/\tau$ by $\varphi = L\Phi$ (where L is the operator of unilateral integral Laplace transformation), it can be shown that corresponding to the complex modulus (4) there is the following distribution function [3]:

$$F(y) = \frac{\sin 2\psi}{\pi(\operatorname{ch} 2\gamma y + \cos 2\psi)},$$

$$y = \ln\frac{\tau}{\tau_0} \qquad F(y)\,dy = -\Phi(s)\,ds, \qquad \psi = \frac{1}{2}\pi\gamma, \qquad (5)$$

where τ_0 is the relaxation time corresponding to the maximum of the distribution function of the logarithms of relaxation time F(g).

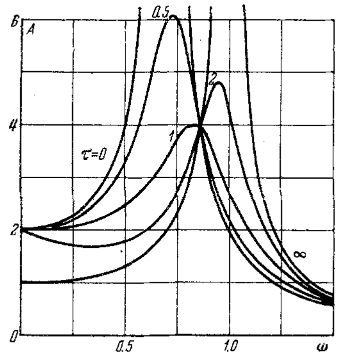

Fig. 2

Relaxations of the type of (4) and (5) are widely used in approximate descriptions of dielectric losses [4]. The phase diagram of the complex elastic modulus $\mu'' = \mu''(\mu')$ describably by (4) is an arc of a circle whose subtended angle is equal to $\pi\gamma$. At $\gamma = 1$, Rabotnov's memory function degenerates into an exponential, and the rheological properties of the system are described by the model of a standard linear body.

Let us consider the dissipative properties of a system describable by a fractionally-exponential memory function. To this end, we substitute in the equation of motion of a univariate oscillator,

$$x^{\cdot\cdot} + \omega_\infty^2 x - (\omega_\infty^2 - \omega_0^2) \int_{-\infty}^{t} f(t-t')\,x(t')\,dt' = Pe^{i\omega t}, \qquad (6)$$

the value $f(t)$ in accordance with (3). Then, taking $x = A \exp i\omega t$, we obtain for the complex amplitude

$$A = P\left[\omega_\infty^2 - \omega^2 - \frac{\omega_\infty^2 - \omega_0^2}{1 + (i\omega\tau)^\gamma}\right]^{-1}. \qquad (7)$$

Hence, assuming $A/P = a \exp(-i\varphi)$, we find the amplitude of the steady-state mode and the phase difference

$$a = \left[\frac{\varkappa^\gamma + \varkappa^{-\gamma} + 2\cos\psi}{R_\infty^2\varkappa^\gamma + R_0^2\varkappa^{-\gamma} + 2R_\infty R_0\cos\psi}\right]^{1/2},$$

$$\operatorname{tg}\varphi = \frac{\Delta R\sin\psi}{R_\infty\varkappa^\gamma + R_0\varkappa^{-\gamma} + \Delta R\cos\psi}$$

$$R_0 = \omega_0^2 - \omega^2, \quad R_\infty = \omega_\infty^2 - \omega^2,$$

$$\Delta R = R_\infty - R_0, \quad \varkappa = \omega\tau. \qquad (8)$$

It can be shown that expressions in (8) are equivalent to the corresponding formulas obtained in [5] by a different method.

Figure 1 shows the resonance amplitudes for $\mu_0/\mu_\infty = 0.5$ and $\gamma = 0.9$, the corresponding data for $\gamma = 1$ (standard linear body), being shown in Fig. 2; the numbers by the curves indicate the values of τ.

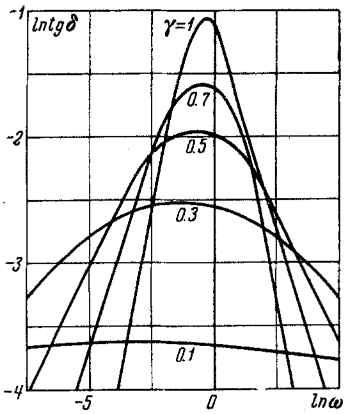

Fig. 3

Thus, the curve $\tau = \infty$ in both cases describes oscillations without energy dissipation at the nonrelaxed natural frequency $\omega_\infty = 1$, while the limiting curve $\tau = 0$ corresponds to the relaxed frequency ω_0. These curves intersect each other at the point

$$\omega_* = (\omega_\infty / \sqrt{2})\,(1 + \mu_0/\mu_\infty)^{1/2}. \qquad (9)$$

The amplitudes at this point for arbitrary τ are given by the expression

$$a_*^2 = R_{\infty*}^{-2}\,\frac{\varkappa_*^{\gamma} + \varkappa_*^{-\gamma} + 2\cos\psi}{\varkappa_*^{\gamma} + \varkappa_*^{-\gamma} - 2\cos\psi}.$$

Hence it will be seen that for a standard linear body there is an optimum amplitude $a_* = R_{\infty*}^{-1}$ which is independent of τ, while no such optimum amplitude exists in the case of arbitrary γ.

Let us determine the internal friction of a medium with Rabotnov's memory function [2] in terms of the mechanical tangent tg δ. The real and imaginary parts of the complex elastic modulus are

$$\mu' = \mu_\infty - \frac{\Delta\mu\,(\varkappa^{-\gamma} + \cos\psi)}{\varkappa^{\gamma} + \varkappa^{-\gamma} + 2\cos\psi},$$

$$\mu'' = \frac{\Delta\mu \sin\psi}{\varkappa^{\gamma} + \varkappa^{-\gamma} + 2\cos\psi}. \qquad (10)$$

Hence

$$\operatorname{tg}\delta = \frac{\Delta\mu\sin\psi}{\mu_\infty\varkappa^{\gamma} + \mu_0\varkappa^{-\gamma} + (\mu_\infty + \mu_0)\cos\psi}. \qquad (11)$$

Consequently, tg δ is determined by the modulus defect, by the diffusion coefficient of the relaxation spectrum γ, and by a dimensionless frequency $\varkappa$. Curves representing the frequency dependence of tg δ for $\mu_0/\mu_\infty = 0.5$ and 0 are reproduced, respectively, in Figs. 3 and 4; numbers by the curves show the values of γ, and $\tau = 1$. At $\gamma = 1$ and $\mu_0 \neq 0$, Eq. (11) describes the internal friction of a standard linear body; at $\gamma = 1$ and $\mu_0 = 0$, this formula is applicable to a Maxwell body. The position of the internal friction peak on the frequency axis depends on the modulus defect and parameter γ; this dependence is described by

$$\Omega\tau = (\mu_0/\mu_\infty)^{1/2\gamma}, \qquad (12)$$

from which it follows that at small γ (broad relaxation spectrum) the frequency Ω is strongly dependent on γ. The height of the internal friction peak is given by

$$\operatorname{tg}\delta_m = \frac{\Delta_\mu \sin\psi}{2 + \Delta_\mu\cos\psi}, \qquad \Delta_\mu \equiv \frac{\mu_\infty - \mu_0}{\sqrt{\mu_\infty\mu_0}}. \qquad (13)$$

It follows that the ratio of the heights of internal friction peaks for the two given values of the modulus defect are weakly dependent on γ, if $(\Delta\mu\cos\psi)/2 \ll 1$. In the case of a standard linear body, Eq. (13) leads to a known expression, tg $\delta_m = \Delta\mu/2$.

Regarding the possibility of using fractionally-exponential memory functions for describing internal friction of solids, it should be pointed out that this method is sufficiently effective if the frequency dependence of the virtual part of the elastic modulus has a symmetrical bell-shaped form. In the case of a substantially asymmetrical distribution function of the logarithms of the relaxation times F(y), which applies, for instance, to polymers in the zone of transition from the highly elastic to the glassy (brittle) state [6], the fractionally-exponential function may be useful in describing the low-frequency asymptotic behavior. At $\mu_0 = 0$, we have from (12)

$$\operatorname{tg}\delta = (\cos\psi + \varkappa^{\gamma})^{-1}\sin\psi. \qquad (14)$$

In the case of $\gamma = 1$, Eq. (14) gives the well-known expression tg $\delta = 1/\omega\tau$ for the internal friction of Maxwell body, widely used in describing the high-temperature background of internal friction of metals [7,8]. The substantial difference between internal friction described by the Maxwell model and by Eq. (14) at $\gamma \neq$ consists in that at $\omega \to 0$, in the former case tg $\delta \to \infty$ and in the latter tg $\delta \to$ tg ψ; this difference is illustrated in Fig. 4.

The high-frequency asymptotic form of (14) gives tg $\delta = (\omega\tau)^{-\gamma} \times \sin\psi$, which—correct to a constant factor—coincides with the empirical formula tg $\delta = (\omega\tau)^{-\gamma}$ postulated in [9] to describe the dislocation background of internal friction of metals. The problem of the possibility of using Eq. (14) in the entire frequency (temperature) range is more complex. In experimental work one usually determines the logarithmic decrement Δ rather than the mechanical loss tangent. When the attenuation is weak tg $\delta = \Delta/\pi$; when, however, the attenuation is strong Δ/π tg δ. Consequently, the absence of a saturation range on experimental curves representing the temperature dependence of Δ/π right up to $\Delta/\pi \sim 1$ cannot be regarded as grounds for asserting that formula (14) is not applicable in the low-frequency (high-temperature) range. It is interesting that in some cases the internal friction measured from the inverse amplitude of forced oscillations under resonance conditions shows a marked reduction in the rate of growth of tg δ near the melting point of a given metal [10].

The authors thank Yu. N. Rabotnov for his helpful comments on the results of this work.

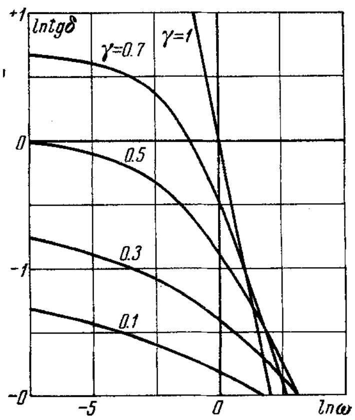

Fig. 4

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18 March 1966 Voronezh